# ARITHMETIC QUOTIENTS OF THE COMPLEX BALL AND A CONJECTURE OF LANG

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## Introduction

Let $F$ be a totally real number field of degree $d$ and ring of integers $\mathfrak{o}$, and let $M$ be a totally imaginary quadratic extension of $F$ with ring of integers $\mathfrak{O}$. Let $G$ be a unitary group over $F$ defined by a hermitian form on $M^{n+1}$ of signature $(n,1)$ at one infinite place $\iota$ and $(n+1,0)$ or $(0,n+1)$ at the others. A subgroup $\Gamma \subset G(F)$ is arithmetic if it is commensurable with $G(\mathfrak{o})$ – the stabilizer in $G(F)$ of $\mathfrak{O}^{n+1}$ – and we will denote by $Y_\Gamma$ the quotient of the $n$-dimensional complex hyperbolic space by the natural action of $\iota(\Gamma) \subset G(F_\iota) = \mathrm{U}(n,1)$. If $F \neq \mathbb{Q}$, then the hermitian form is anisotropic and $Y_\Gamma$ is a projective variety defined over a number field (see Proposition 1.2).

A projective variety $X$ over $\mathbb{C}$ is said to be Mordellic if it has only a finite number of rational points in every finitely generated field extension of $\mathbb{Q}$ over which $X$ is defined. Lang conjectured in [L, Conjecture VIII.1.2] that $X$ is Mordellic if and only if the corresponding analytic space $X(\mathbb{C})$ is hyperbolic, meaning that any holomorphic map $\mathbb{C} \to X(\mathbb{C})$ is constant, which by Brody [B] is equivalent to requiring the Kobayashi semi-distance on $X(\mathbb{C})$ to be a metric. It is a consequence of a conjecture of Ullmo (see [U, Conjecture 2.1]) that a projective variety $X$ defined over a number field $k$ is Mordellic if it is arithmetically Mordellic, meaning that it has only a finite number of rational points in every finite extension of $k$.

Our first result establishes that many arithmetic compact surfaces previously only known to be arithmetically Mordellic by [U, Théorème 3.2] are in fact Mordellic. To state it precisely we need to fix a Hecke character $\lambda$ of $M$ as in Definition 3.1. The existence of such characters is known (see Lemma 3.5). Denote by $\mathfrak{C}$ the conductor of $\lambda$ and, if the extension $M/F$ is everywhere unramified, we multiply $\mathfrak{C}$ by any prime $\mathfrak{q}$ of $F$ which does not split in $M$. Moreover, fix an auxiliary prime $\mathfrak{p}$ of $F$ which splits in $M$ and is relatively prime to $\mathfrak{C}$. Finally, for every ideal $\mathfrak{N} \subset \mathfrak{O}$ we consider the standard congruence subgroups $\Gamma_0(\mathfrak{N})$, $\Gamma_1(\mathfrak{N})$ and $\Gamma(\mathfrak{N})$ of $G(F)$ (see Definition 1.3).

**Theorem 0.1.** *Let $n=2$ and $G$ over $F$ as above. Then for every choice of $(\mathfrak{C},\mathfrak{p})$, and for any torsion free subgroup $\Gamma \subset \Gamma_1(\mathfrak{C}) \cap \Gamma_0(\mathfrak{p})$ of finite index, $Y_\Gamma$ is Mordellic.*

A consequence of this is that for any arithmetic subgroup $\Gamma \subset G(F)$ there exists a finite explicit cover of $Y_\Gamma$ which is Mordellic. Note also that even though the theorem only concerns arithmetic subgroups, because $F$ and $M$ can vary, it can be applied to infinitely many pairwise non-commensurable cocompact discrete subgroups in $\mathrm{U}(2,1)$. In order to apply our method to the analogous case of a unitary group $G''$ defined by a division algebra of dimension 9 over $M$ with an involution of the second kind, one would need to find a (cocompact) arithmetic subgroup $\Gamma \subset G''(F)$ such that the Albanese of $Y_\Gamma$ is non-zero. This is an open question for any $G''$ since, in contrast to our case, it is known by Rogawski [R1] that the Albanese of $Y_\Gamma$ is zero for any congruence subgroup $\Gamma \subset G''(F)$.

While Ullmo's approach uses the Shafarevich conjecture, ours is based instead on the Mordell-Lang conjecture proved by Faltings [F2] and on the key Proposition 3.6, which we hope is of independent interest.

Consider now the case when the hermitian form is isotropic, which necessarily implies that $F = \mathbb{Q}$ and $M$ is imaginary quadratic. Then $Y_\Gamma$ is not compact and, for $\Gamma$ arithmetic, we denote by $Y_\Gamma^*$ the Baily-Borel compactification which is a normal, projective variety of dimension $n$. A smooth toroidal compactification $X_\Gamma$ of $Y_\Gamma$ can be defined over a number field (see [F3]), and it is not hyperbolic even if $\Gamma$ is torsion-free; for example, if $n = 2$, then $X_\Gamma$ is a union of $Y_\Gamma$ with a finite number of elliptic curves – one above each cusp of $Y_\Gamma^*$. However, by a result of Tai and Mumford [Mu, §4], $X_\Gamma$ is of general type for $\Gamma$ sufficiently small. The *Bombieri-Lang conjecture* asserts then that the points of $X_\Gamma$ over any finitely generated field extension of $\mathbb{Q}$ over which $X_\Gamma$ is defined are not Zariski dense. We prove this in Proposition 4.3 which allows us to solve an alternative of Ullmo and Yafaev [UY] regarding the Lang locus of $Y_\Gamma^*$.

**Theorem 0.2.** *For all $\Gamma \subset G(\mathbb{Q})$ arithmetic and sufficiently small, $Y_\Gamma^*$ is arithmetically Mordellic.*

Keeping the assumption that $M$ is imaginary quadratic, say of fundamental discriminant $-D$, let us now suppose in addition that $n = 2$. The corresponding locally symmetric spaces $Y_\Gamma$ are called Picard modular surfaces. We state here our main theorem.

**Theorem 0.3.** *Let* $\mathfrak{D} = \begin{cases} 3\mathfrak{O} & \text{, if } D = 3, \\ \sqrt{-D}\mathfrak{O} & \text{, if } D \neq 3 \text{ is odd,} \\ 2\sqrt{-D}\mathfrak{O} & \text{, if } 8 \text{ divides } D, \\ \sqrt{-D}\mathfrak{P}_2 & \text{, otherwise, where } \mathfrak{P}_2 \text{ is the prime of } M \text{ above } 2. \end{cases}$

(i) *Let* $\Gamma = \begin{cases} \Gamma_1(\mathfrak{D}) & \text{, if } D \notin \{3,4,7,8,11,15,19,20,23,24,31,39,43,47,67,71,163\}, \\ \Gamma(\mathfrak{D}) & \text{, if } D \in \{8,15,20,23,24,31,39,47,71\}, \\ \Gamma(\mathfrak{D}^2) & \text{, if } D \in \{3,4,7,11,19,43,67,163\}. \end{cases}$

*Then* $Y^*_\Gamma$ *is Mordellic, while* $X_\Gamma$ *is a minimal surface of general type.*

(ii) *Let* $N > 2$ *be a prime inert in* $M$ *and not equal to* $3$ *when* $D = 4$. *Then* $Y^*_{\Gamma(N)\cap\Gamma_1(\mathfrak{D})}$ *is Mordellic, while* $X_{\Gamma(N)\cap\Gamma_1(\mathfrak{D})}$ *is a minimal surface of general type.*

The cases $D \in \{15, 20, 23, 24, 31, 39, 47, 71\}$ in (i) of Theorem 0.3 depend on a preprint of Džambić [D] which is being considered for publication elsewhere (see the proof for details).

At the heart of our proof stand some arithmetical computations using certain key theorems of Rogawski [R1, R2]. They yield, for each imaginary quadratic field $M$, an explicit congruence subgroup $\Gamma$ such that the smooth compactification $X_\Gamma$ does not admit a dominant map to its Albanese variety. A geometric ingredient of the proof is a result of Holzapfel *et al* that $X_\Gamma$ is of general type, though not hyperbolic, implying by a theorem of Nadel [N] that any curve of genus $\leq 1$ on it is contained in the compactifying divisor.

If there is anything new in our approach, it lies in the systematic use of the modern theory of automorphic representations in Diophantine geometry.

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## 1. Basics: lattices, general type and neatness

For any integer $n > 1$, let $\mathcal{H}^n_\mathbb{C}$ be the $n$-dimensional complex hyperbolic space, represented by the unit ball in $\mathbb{C}^n$ equipped with the Bergman metric of constant holomorphic sectional curvature $-4/(n+1)$, on which the real Lie group $\mathrm{U}(n,1)$ acts in a natural way.

Given a lattice $\Gamma \subset \mathrm{U}(n,1)$ we denote by $\bar\Gamma = \Gamma/\Gamma \cap \mathrm{U}(1)$ its image in the adjoint group $\mathrm{PU}(n,1) = \mathrm{U}(n,1)/\mathrm{U}(1)$, where $\mathrm{U}(1)$ is centrally embedded in $\mathrm{U}(n,1)$. Conversely any lattice $\bar\Gamma \subset \mathrm{PU}(n,1) = \mathrm{PSU}(n,1)$ is the image of a lattice in $\mathrm{U}(n,1)$, namely the lattice $\mathrm{U}(1)\bar\Gamma \cap \mathrm{SU}(n,1)$. We consider the quotient $Y_\Gamma = Y_{\bar\Gamma} = \bar\Gamma\backslash\mathcal{H}^n_\mathbb{C}$.

**Lemma 1.1.** *Let* $\Gamma$ *be a lattice in* $\mathrm{U}(n,1)$.

(i) *The analytic variety* $Y_\Gamma$ *is an orbifold and one has the following implications:*

$$\Gamma \textit{ neat} \Rightarrow \Gamma \textit{ torsion-free} \Rightarrow \bar{\Gamma} \textit{ torsion-free} \Rightarrow Y_\Gamma \textit{ is a hyperbolic manifold.}$$

(ii) *Assume that* $\bar{\Gamma}$ *is torsion-free. Then the natural projection* $\mathcal{H}_{\mathbb{C}}^n \to Y_\Gamma$ *is an etale covering with deck transformation group* $\bar{\Gamma}$.

*Proof.* The stabilizer in $\mathrm{U}(n,1)$ of any point of $\mathcal{H}_{\mathbb{C}}^n$ is a compact group, hence its intersection with the discrete subgroup $\Gamma$ is finite, showing that $Y_\Gamma$ is an orbifold.

Recall that $\Gamma$ is neat if the subgroup of $\mathbb{C}^\times$ generated by the eigenvalues of any $\gamma \in \Gamma$ is torsion-free. In particular $\Gamma$ is torsion-free. Since $\Gamma \cap \mathrm{U}(1)$ is finite, this implies that $\bar{\Gamma}$ is torsion-free too. Under the latter assumption, $\Gamma \cap \mathrm{U}(1)$ acts trivially on $\mathcal{H}_{\mathbb{C}}^n$, and $\bar{\Gamma}$ acts freely and properly discontinuously on it, hence $Y_\Gamma$ is a manifold. Since $\mathcal{H}_{\mathbb{C}}^n$ is simply connected, it is a universal covering space of $Y_\Gamma$ with group $\bar{\Gamma}$. Hence any holomorphic map from $\mathbb{C}$ to $Y_\Gamma$ lifts to a holomorphic map from $\mathbb{C}$ to $\mathcal{H}_{\mathbb{C}}^n$ which must be constant because $\mathcal{H}_{\mathbb{C}}^n$ has negative curvature. Thus $Y_\Gamma$ is hyperbolic. □

Deligne's classification [De] of Shimura varieties implies, when $\Gamma$ is a congruence subgroup, that $Y_\Gamma$ admits an embedding in a Shimura variety. Hence, by Shimura's theory of canonical models, $Y_\Gamma$ can be defined over a finite abelian extension of the reflex field $M$.

We claim that this is also true for $\Gamma$ arithmetic, when sufficiently small. Indeed any such $Y_\Gamma$ is a finite unramified cover of a congruence quotient $Y_{\Gamma'}$ which we have seen is defined over a number field. By Grothendieck, the finite index subgroup $\bar{\Gamma}$ of the topological fundamental group $\bar{\Gamma}'$ of $Y_{\Gamma'}(\mathbb{C})$ gives rise to a finite index subgroup of the algebraic fundamental group of $Y_{\Gamma'}$, yielding a finite algebraic (etale) map from a model of $Y_\Gamma$ to $Y_{\Gamma'}$.

In the cocompact case, this remains true even when $\Gamma$ is not arithmetic.

**Proposition 1.2.** *Assume that* $\bar{\Gamma}$ *is cocompact and torsion-free. Then the projective variety* $Y_{\bar{\Gamma}}$ *is of general type and can be defined over a number field.*

*Proof.* The existence of the positive Bergman metric on $\mathcal{H}_{\mathbb{C}}^n$ implies by the Kodaira embedding theorem that any quotient by a free action such as $Y_{\bar{\Gamma}}$ has ample canonical bundle, which results in $Y_{\bar{\Gamma}}$ being of general type; it even implies that any subvariety of $Y_{\bar{\Gamma}}$ is of general type. For surfaces one may alternately use the hyperbolicity of $Y_{\bar{\Gamma}}$ to rule out all the cases in the Enriques-Kodaira classification where the Kodaira dimension is less than 2, thus showing that $Y_{\bar{\Gamma}}$ is of general type.

Calabi and Vesentini [CV] have proved that $Y_{\bar{\Gamma}}$ is locally rigid, hence by Shimura [S1] it can be defined over a number field.

In order to highlight the importance of rigidity of compact ball quotients, we provide a short second proof when $n = 2$ and $\Gamma$ is arithmetic, based on Yau's algebro-geometric characterization of compact Kähler surfaces covered by $\mathcal{H}^2_{\mathbb{C}}$. Since $Y_{\bar\Gamma}$ has an ample canonical bundle it can be embedded in some projective space, hence is algebraic over $\mathbb{C}$ by Chow. Since $Y_{\bar\Gamma}$ is uniformized by $\mathcal{H}^2_{\mathbb{C}}$, the Chern numbers $c_1, c_2$ of its complex tangent bundle satisfy the relation $c_1^2 = 3c_2$. Since everything can be defined algebraically, for any automorphism $\sigma$ of $\mathbb{C}$, the variety $Y^\sigma_{\bar\Gamma}$ also has ample canonical bundle and $c_1^{\sigma 2} = 3c_2^\sigma$. By a famous result of Yau [Y, Theorem 4], this is equivalent to the fact that $Y^\sigma_{\bar\Gamma}$ may be realized as $\bar\Gamma^\sigma \backslash \mathcal{H}^2_{\mathbb{C}}$ for some cocompact torsion-free lattice $\bar\Gamma^\sigma$.

Since $\bar\Gamma$ is arithmetic, it has infinite index in its commensurator in $\mathrm{PU}(2,1)$, denoted by $\mathrm{Comm}(\bar\Gamma)$. For every element $g \in \mathrm{Comm}(\bar\Gamma)$ there is a Hecke correspondence

$$Y_{\bar\Gamma} \leftarrow Y_{\bar\Gamma \cap g^{-1}\bar\Gamma g} \xrightarrow[g\cdot]{\sim} Y_{g\bar\Gamma g^{-1}\cap\bar\Gamma} \rightarrow Y_{\bar\Gamma} \tag{1}$$

and the correspondences for $g$ and $g'$ differ by an isomorphism $Y_{g\bar\Gamma g^{-1}\cap\bar\Gamma} \xrightarrow{\sim} Y_{g'\bar\Gamma g'^{-1}\cap\bar\Gamma}$ over $Y_{\bar\Gamma}$ if and only if $g' \in \bar\Gamma g$. By Chow (1) is defined algebraically, hence yields a correspondence on $Y^\sigma_{\bar\Gamma} = Y_{\bar\Gamma^\sigma}$:

$$Y_{\bar\Gamma^\sigma} \leftarrow Y_{\bar\Gamma_1} \xrightarrow{\sim} Y_{\bar\Gamma_2} \rightarrow Y_{\bar\Gamma^\sigma},$$

for some finite index subgroups $\bar\Gamma_1$ and $\bar\Gamma_2$ of $\bar\Gamma^\sigma$. By the universal property of the covering space $\mathcal{H}^2_{\mathbb{C}}$, the middle isomorphism is given by an element of $g_\sigma \in \mathrm{PU}(2,1) \simeq \mathrm{Aut}(\mathcal{H}^2_{\mathbb{C}})$. Since $\mathrm{Aut}(\mathcal{H}^2_{\mathbb{C}}/Y_{\bar\Gamma_i}) = \bar\Gamma_i$ $(i = 1,2)$, it easily follows that $\bar\Gamma_2 = g_\sigma \bar\Gamma_1 g_\sigma^{-1}$, and by applying $\sigma^{-1}$ one sees that $\bar\Gamma_1 = \bar\Gamma^\sigma \cap g_\sigma^{-1}\bar\Gamma^\sigma g_\sigma$. It follows that $g_\sigma \in \mathrm{Comm}_G(\Gamma^\sigma)$ and one can check that $g'_\sigma \in \bar\Gamma^\sigma g_\sigma$ if and only if $g' \in \bar\Gamma g$. Therefore $\mathrm{Comm}(\bar\Gamma^\sigma)/\bar\Gamma^\sigma \simeq \mathrm{Comm}(\bar\Gamma)/\bar\Gamma$ is infinite too, which by a major theorem of Margulis implies that $\bar\Gamma^\sigma$ is arithmetic, providing an alternative proof of a result of Kazhdan.

Thus $\mathrm{Aut}(\mathbb{C})$ acts on the set of isomorphism classes of cocompact arithmetic quotients $Y_{\bar\Gamma}$, or equivalently, on the set of equivalence classes of cocompact arithmetic subgroups $\bar\Gamma$ (up to conjugation by an element of $\mathrm{PU}(2,1)$). The latter set is countable for the following reason. The group $\mathrm{U}(2,1)$ has only countably many $\mathbb{Q}$-forms, classified by central simple algebras of dimension 9 over $M$, endowed with an involution of a second kind and verifying some conditions at infinity (see [PR, pp. 87-88]). Moreover, there are only countably many arithmetic subgroups for a given $\mathbb{Q}$-form, since those are all finitely generated and contained in their common commensurator, which is countable.

Finally, by [G, Corollary 2.13], the fact that $Y_{\bar\Gamma}$ has a countable orbit under the action of $\mathrm{Aut}(\mathbb{C})$ is equivalent to $Y_{\bar\Gamma}$ being defined over a number field. □

It is a well known fact that any orbifold admits a finite cover which is a manifold. In view of Lemma 1.1, the two lemmas below provide such covers explicitly for arithmetic quotients.

**Definition 1.3.** For every ideal $\mathfrak{N} \subset \mathfrak{O}$ we define the congruence subgroup $\Gamma(\mathfrak{N})$ (resp. $\Gamma_0(\mathfrak{N})$, resp. $\Gamma_1(\mathfrak{N})$) as the kernel (resp. the inverse image of upper triangular, resp. upper unipotent, matrices) of the composite homomorphism:

$$G(\mathfrak{o}) \hookrightarrow \mathrm{GL}(n+1, \mathfrak{O}) \to \mathrm{GL}(n+1, \mathfrak{O}/\mathfrak{N}).$$

The following lemma is well-known (see [H1, Lemma 4.3]).

**Lemma 1.4.** *For any integer $N > 2$ the group $\Gamma(N)$ is neat.*

**Lemma 1.5.** *Suppose that $n = 2$ and that $M$ is an imaginary quadratic field of fundamental discriminant $-D \notin \{-3, -4, -7, -8, -24\}$. Then $\Gamma_1(\sqrt{-D}\mathfrak{O})$ is neat.*

*Proof.* Suppose that the subgroup of $\mathbb{C}^\times$ generated by the eigenvalues of some $\gamma \in \Gamma_1(\sqrt{-D}\mathfrak{O})$ contains a non-trivial root of unity. Note first that $\det(\gamma) \in \mathfrak{O}^\times \cap (1 + \sqrt{-D}\mathfrak{O}) = \{1\}$.

If $\gamma$ is elliptic then it is necessarily of finite order. Otherwise $\gamma$ fixes a boundary point of $\mathcal{H}^2_{\mathbb{C}} \subset \mathbb{P}^2(\mathbb{C})$ and is therefore conjugated in $\mathrm{GL}(3, \mathbb{C})$ to a matrix of the form $\left(\begin{smallmatrix} \bar{\alpha} & * & * \\ 0 & \beta & * \\ 0 & 0 & \alpha^{-1} \end{smallmatrix}\right)$, where $\beta$ is necessarily a root of unity. If $\beta = 1$, then $\det(\gamma) = 1$ implies that $\alpha \in \mathbb{R}$, leading to $\alpha = -1$. Hence, in all cases, one may assume $\gamma$ has a non-trivial root of unity $\zeta$ as an eigenvalue.

By the Cayley-Hamilton theorem we have $[M(\zeta) : M] \leq 3$ and since $D \neq 7$ we may assume (after possibly raising $\gamma$ to some power) that $\zeta$ has order 2 or 3. By the congruence condition, each prime $p$ dividing $D$ has to divide also the norm of $\zeta - 1$, hence $D$ can be divisible only by the primes 2 or 3. Thus $D \in \{3, 4, 8, 24\}$, leading to a contradiction. $\square$

## 2. Irregularity of arithmetic varieties

Let $z \mapsto \bar{z}$ be the non-trivial automorphism of $M/F$ and let $\omega$ be the quadratic character of $M/F$, viewed as a Hecke character of $F$. Put $M^1 = \{z \in M^\times \mid z\bar{z} = 1\}$, which we will view as an algebraic torus over $F$ and denote by $\mathbb{A}^1_M$ its $\mathbb{A}_F$-points.

We denote by $q(X)$ the *irregularity* of $X$, given by the dimension of $\mathrm{H}^0(X, \Omega^1_X)$.

2.1. **Automorphic forms contributing to the irregularity.** Fix a maximal compact subgroup $K_\infty \simeq (\mathrm{U}(n)\times\mathrm{U}(1))\times\mathrm{U}(n+1)^{d-1}$ of the real linear Lie group $G_\infty = G(F\otimes_{\mathbb{Q}}\mathbb{R}) \simeq \mathrm{U}(n,1) \times \mathrm{U}(n+1)^{d-1}$. Let $\Gamma \subset G(F)$ be a lattice such that $\bar{\Gamma}$ is torsion free.

Since $Y_\Gamma$ is the Eilenberg-MacLane space of $\overline{\Gamma}$, there is a decomposition:

$$\mathrm{H}^1(Y_\Gamma, \mathbb{C}) \simeq \mathrm{H}^1(\bar{\Gamma}, \mathbb{C}) \simeq \bigoplus_{\pi_\infty} \mathrm{H}^1(\mathrm{Lie}(G_\infty), K_\infty; \pi_\infty)^{\oplus m(\pi_\infty, \Gamma)}, \tag{2}$$

where $\pi_\infty$ runs over irreducible unitary representations of $G_\infty$ occurring in the discrete spectrum of $\mathrm{L}^2(\Gamma\backslash G_\infty)$ with multiplicity $m(\pi_\infty, \Gamma)$, and $\mathrm{H}^*(\mathrm{Lie}(G_\infty), K_\infty; \pi_\infty)$ is the relative Lie algebra cohomology. When $\Gamma$ is cocompact, the entire $\mathrm{L}^2$-spectrum is discrete and this decomposition follows from [BW, XIII]. When $\Gamma$ is non-cocompact, one gets by [BC, §4.4-4.5] such a decomposition, but only for the $\mathrm{L}^2$-cohomology of $Y_\Gamma$. However, one knows (see [MR, §1]) that $\mathrm{H}^1(Y_\Gamma, \mathbb{C})$ is isomorphic to the middle intersection cohomology (in degree 1) of $Y_\Gamma^*$, which is in turn isomorphic to the $\mathrm{L}^2$-cohomology (in degree 1) of $Y_\Gamma$.

By [BW, VI.4.11] there are exactly two irreducible non-tempered unitary representations of $\mathrm{SU}(n,1)$ with trivial central character, denoted $J_{1,0}$ and $J_{0,1}$, each of whose relative Lie algebra cohomology in degree 1 does not vanish and is in fact one dimensional. Since $\mathrm{U}(n,1)$ is the product of its center with $\mathrm{SU}(n,1)$, $J_{1,0}$ and $J_{0,1}$ can be uniquely extended to representations $\pi^+$ and $\pi^-$, say, of $\mathrm{U}(n,1)$ with trivial central characters (when $n=2$ those are the representations $J^\pm$ from [R1, p.178]). It follows that at the distinguished Archimedean place $\iota$, where $G(F_\iota) = \mathrm{U}(n,1)$, we have

$$\mathrm{H}^1(\mathrm{Lie}(\mathrm{U}(n,1)), \mathrm{U}(n) \times \mathrm{U}(1); \pi_\iota) = \begin{cases} \mathbb{C} \text{ , if } \pi_\iota = \pi^\pm, \\ 0 \text{ , otherwise.} \end{cases}$$

Moreover the only irreducible unitary representation with non-zero relative Lie algebra cohomology in degree 0 is the trivial representation $\mathbb{1}$, which does not contribute in degree 1; in particular $\pi^\pm \neq \mathbb{1}$. This allows us to deduce from (2) the following formula

$$\dim_{\mathbb{C}} \mathrm{H}^1(Y_\Gamma, \mathbb{C}) = m(\pi^+ \otimes \mathbb{1}^{\otimes d-1}, \Gamma) + m(\pi^- \otimes \mathbb{1}^{\otimes d-1}, \Gamma), \tag{3}$$

where $\pi^\pm$ are viewed as representations of $G(F_\iota) = \mathrm{U}(n,1)$ and $\mathbb{1}^{\otimes d-1}$ denotes the trivial representation of $\mathrm{U}(n+1)^{d-1}$.

By [MR, §1], $\mathrm{H}^1(Y_\Gamma, \mathbb{C})$ is isomorphic to $\mathrm{H}^1(X_\Gamma, \mathbb{C})$, hence admits a pure Hodge structure of weight 1 and its dimension is given by $2q(X_\Gamma)$. In particular, the natural map $\mathrm{H}^0(X_\Gamma, \Omega^1_{X_\Gamma}) \to \mathrm{H}^0(Y_\Gamma, \Omega^1_{Y_\Gamma})$ is an isomorphism, *i.e.*,

$$q(Y_\Gamma) = q(X_\Gamma). \tag{4}$$

It is known that $\pi^+ \otimes \mathbb{1}^{\otimes d-1}$ (resp. $\pi^- \otimes \mathbb{1}^{\otimes d-1}$) contributes to $\mathrm{H}^0(Y_\Gamma, \Omega^1_{Y_\Gamma})$ (resp. $\mathrm{H}^1(Y_\Gamma, \Omega^0_{Y_\Gamma})$). Since the latter two groups have the same dimension, it follows from (3) that

$$q(Y_\Gamma) = m(\pi^+ \otimes \mathbb{1}^{\otimes d-1}, \Gamma) = m(\pi^- \otimes \mathbb{1}^{\otimes d-1}, \Gamma). \tag{5}$$

We will now focus on the case when $\Gamma$ is a congruence subgroup and switch to the adelic setting which is better suited for computing the irregularity. For any open compact subgroup $K$ of $G(\mathbb{A}_{F,f})$, where $\mathbb{A}_{F,f}$ denotes the ring of finite adeles of $F$, we consider the adelic quotient

$$Y_K = G(F)\backslash G(\mathbb{A}_F)/K K_\infty. \tag{6}$$

Let $G^1 = \ker(\det : G \to M^1)$ be the derived group of $G$. Since $G^1$ is simply connected and $G^1_\infty$ is non-compact, $G^1(F)$ is dense in $G^1(\mathbb{A}_{F,f})$ by strong approximation (see [PR, Theorem 7.12]). It follows that the group of connected components of $Y_K$ is isomorphic to the idele class group:

$$\pi_0(Y_K) \simeq \mathbb{A}_M^1/M^1 \det(K) M_\infty^1. \tag{7}$$

To describe each connected component of $Y_K$, choose $t_i \in G(\mathbb{A}_{F,f})$, $1 \leq i \leq h$, such that $(\det(t_i))_{1\leq i\leq h}$ forms a complete set of representatives of $\mathbb{A}_M^1/M^1 \det(K) M_\infty^1$, and let $\Gamma_i = G(F) \cap t_i K t_i^{-1} G_\infty$. Then

$$G(F)\backslash G(\mathbb{A}_F)/K = \coprod_{i=1}^{h} \Gamma_i\backslash G_\infty \text{ and } Y_K = \coprod_{i=1}^{h} Y_{\Gamma_i}. \tag{8}$$

Therefore (2) and (5) can be rewritten as:

$$\mathrm{H}^1(Y_K, \mathbb{C}) \simeq \bigoplus_{\pi=\pi_\infty\otimes\pi_f} \left(\mathrm{H}^1(\mathrm{Lie}(G_\infty), K_\infty; \pi_\infty) \otimes \pi_f^K\right)^{\oplus m(\pi)} \text{ and} \tag{9}$$

$$q(Y_K) = \sum_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty=\pi_\iota\otimes 1^{\otimes d-1}, \pi_\iota\simeq\pi^+}} m(\pi) \dim(\pi_f^K) = \sum_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty=\pi_\iota\otimes 1^{\otimes d-1}, \pi_\iota\simeq\pi^-}} m(\pi) \dim(\pi_f^K), \tag{10}$$

where $\pi$ runs over all automorphic representation of $G(\mathbb{A}_F)$ occurring discretely, with multiplicity $m(\pi)$, in $\mathrm{L}^2(G(F)\backslash G(\mathbb{A}_F))$.

2.2. **Irregularity growth.** Non-vanishing of $q(Y_\Gamma)$ for sufficiently small congruence subgroups is known by a theorem of Shimura [S2, Theorem 8.1], extending earlier works of Kazhdan and Borel-Wallach [BW, VIII]. Our Diophantine results require however the stronger assumption that $q(Y_\Gamma) > n$, which we establish as a corollary of the next proposition.

**Proposition 2.1.** *For every open compact subgroup $K \subset G(\mathbb{A}_{F,f})$ such that $q(Y_K) \neq 0$ there exist infinitely many primes $\mathfrak{p}$ of $F$ for which one can find an explicit finite index subgroup $K' \subset K$ differing form $K$ only at $\mathfrak{p}$, such that $\pi_0(Y_{K'}) = \pi_0(Y_K)$ and $q(Y_{K'}) > q(Y_K)$.*

*Proof.* Since $q(Y_K) \neq 0$ by assumption, formula (10) implies that there exists an automorphic representation $\pi$ with $\pi_\infty = \pi_\iota \otimes \mathbb{1}^{\otimes d-1}$, $\pi_\iota \simeq \pi^+$, such that $m(\pi) \neq 0$ and $\pi_f^K \neq 0$.

Let $\mathfrak{p}$ be a prime of $F$ which splits in $M$, so that $G(F_\mathfrak{p}) = \mathrm{GL}(n+1, F_\mathfrak{p})$. Assume that $K = K_\mathfrak{p}^0 \times K^{(\mathfrak{p})}$ where $K_\mathfrak{p}^0 = \mathrm{GL}(n+1, \mathfrak{o}_\mathfrak{p})$ is the standard maximal compact subgroup of $G(F_\mathfrak{p})$ and $K^{(\mathfrak{p})}$ is the part of $K$ away from $\mathfrak{p}$. In particular $\pi_\mathfrak{p}$ is unramified. Moreover $\pi_\mathfrak{p}$ is a unitary representation, since it is a local component of an automorphic representation. By the main result of [T], $\pi_\mathfrak{p}$ is then the full induced representation of $\mathrm{GL}(n+1, F_\mathfrak{p})$ from an unramified character $\mu$ of a parabolic subgroup $P(F_\mathfrak{p})$.

We claim that, in our case, $P$ is a proper parabolic subgroup. Otherwise $\pi_{\mathfrak{p}}$ will be one dimensional, hence $G^1(F_{\mathfrak{p}})$ will act trivially. Since by strong approximation $G^1(F)G^1(F_{\mathfrak{p}})$ is dense in $G^1(\mathbb{A}_F)$, the latter will act trivially on any smooth vector in $\pi$, contradicting the fact that $\pi_\iota \not\simeq \mathbb{1}$.

Let $\mathbb{F}_q = \mathfrak{o}/\mathfrak{p}$ be the residue field of $F_{\mathfrak{p}}$ and denote by $P(\mathbb{F}_q)$ the corresponding parabolic subgroup of $G(\mathbb{F}_q)$. Let $K_{0,P}(\mathfrak{p})$ is the inverse image of $P(\mathbb{F}_q)$ under the reduction modulo $\mathfrak{p}$ homomorphism $\mathrm{GL}(n+1, \mathfrak{o}_{\mathfrak{p}}) \to \mathrm{GL}(n+1, \mathbb{F}_q)$.

Consider $K' = K_{0,P}(\mathfrak{p})K^{(\mathfrak{p})}$. Since $\det(K') = \det(K)$, (7) implies that $\pi_0(Y_{K'}) = \pi_0(Y_K)$. Moreover, formula (10) implies that

$$q(Y_{K'}) \geq q(Y_K) + \dim(\pi_{\mathfrak{p}}^{K_{0,P}(\mathfrak{p})}) - \dim(\pi_{\mathfrak{p}}^{K_{\mathfrak{p}}^0}) = q(Y_K) + \dim(\pi_{\mathfrak{p}}^{K_{0,P}(\mathfrak{p})}) - 1,$$

hence it suffices to show that the $K_{0,P}(\mathfrak{p})$-invariants in $\pi_{\mathfrak{p}}$ form at least a 2-dimensional space. We claim that we even have

$$\dim(\pi_{\mathfrak{p}}^{K_{0,P}(\mathfrak{p})}) \geq n+1. \tag{11}$$

Indeed, since $\mu$ is unramified, its restriction to $P \cap K_{\mathfrak{p}}^0$ is trivial. Therefore by the Iwasawa decomposition $G(F_{\mathfrak{p}}) = P(F_{\mathfrak{p}}) \cdot K_{\mathfrak{p}}^0$ the restriction of $\pi_{\mathfrak{p}}$ to $K_{\mathfrak{p}}^0$ is isomorphic to $\mathrm{Ind}_{P(F_{\mathfrak{p}})\cap K_{\mathfrak{p}}^0}^{K_{\mathfrak{p}}^0}(\mathbb{1})$.

The subspace of $K_{0,P}(\mathfrak{p})$-invariant vectors in $\mathrm{Ind}_{P(F_{\mathfrak{p}})\cap K_{\mathfrak{p}}^0}^{K_{\mathfrak{p}}^0}(\mathbb{1})$ identifies naturally with the space of $\mathbb{C}$-valued functions on the set:

$$\left(P(F_{\mathfrak{p}}) \cap K_{\mathfrak{p}}^0\right) \backslash K_{\mathfrak{p}}^0 / K_{0,P}(\mathfrak{p}) \simeq P(\mathbb{F}_q) \backslash G(\mathbb{F}_q) / P(\mathbb{F}_q).$$

and the number of such double cosets is the number of double cosets of the Weyl group of $G$ relative to the subgroup attached to $P$. We may assume, for getting a lower bound, that $P$ is maximal (and proper). The smallest number appears for $P$ of type $(n, 1)$ and it is $n+1$. The claim follows. $\square$

**Corollary 2.2.** *For every arithmetic subgroup $\Gamma \subset G(F)$ there exists an explicit subgroup $\Gamma'$ of finite index in $\Gamma$ such that $q(Y_{\Gamma'}) > q(Y_\Gamma)$.*

*Proof.* By [S2, Theorem 8.1] there exists an open compact subgroup $K$ of $G(\mathbb{A}_{F,f})$ such that $q(Y_K) \neq 0$ (one might take the principal congruence subgroup of level $2N$, which is included in the index 4 subgroup of the principal congruence subgroup of level $N$ considered by Shimura). Denote by $h$ the cardinality of $\pi_0(Y_K)$. Applying recursively Proposition 2.1 yields a finite index subgroup $K' \subset K$, such that $\pi_0(Y_{K'}) = \pi_0(Y_K)$ and

$$q(Y_{K'}) > h \cdot q(Y_\Gamma).$$

Write $Y_{K'} = \coprod_{i=1}^h Y_{\Gamma'_i}$ as in (8), and let $\Gamma' = \cap_{i=1}^h \Gamma'_i$. Since the irregularity cannot decrease by going to a finite cover, one has:

$$q(Y_{\Gamma'\cap\Gamma}) \geq q(Y_{\Gamma'}) \geq \max_{1\leq i\leq h} q(Y_{\Gamma'_i}) \geq \frac{1}{h}\sum_{i=1}^h q(Y_{\Gamma'_i}) = \frac{1}{h} q(Y_{K'}) > q(Y_\Gamma).$$

□

One can simplify the final step of the proof above and use any $\Gamma'_i$ instead of $\cap_{i=1}^h \Gamma'_i$, since Shimura's theory of canonical models implies that the connected components of $Y_{K'}$ are all Galois conjugates, hence share the same irregularity.

## 3. Irregularity of arithmetic surfaces

The positivity of $q(Y_\Gamma)$ is an essential ingredient in the proof of our Diophantine results.

The starting point for the arithmetic application of this paper was our knowledge that Rogawski's classification [R1, R2] of cohomological automorphic forms on $G$, combined with some local representation theory, would allow us to compute $q(Y_\Gamma)$ precisely and show that it does not vanish for some explicit congruence subgroups $\Gamma$. Marshall [Ma] gives sharp asymptotic bounds for $q(Y_\Gamma)$ when $\Gamma$ shrinks, also by using Rogawski's theory.

In this section we assume that $n = 2$.

3.1. **Rogawski's theory.** Rogawski [R1, R2] gives an explicit description, in terms of global Arthur packets, of the automorphic representations $\pi$ of $G(\mathbb{A}_F)$ occurring in (10), which we will now present.

Let $T$ denote the maximal torus of the standard upper-triangular Borel subgroup $B$ of $G$.

Let $G'$ denote the quasi-split unitary group associated to $M/F$, so that $G$ is an inner form of $G'$. Note that $G_v \simeq G'_v$ for any finite place $v$ and that $G \simeq G'$ only for $d = 1$.

Let $\lambda$ be a unitary Hecke character of $M$ whose restriction to $F$ is $\omega$, and let $\nu$ be a unitary character of $\mathbb{A}_M^1/M^1$.

At a place $v$ of $F$ which does not split in $M$, which includes any Archimedean $v$, the local Arthur packet $\Pi'(\lambda_v, \nu_v)$ consists of a square-integrable representation $\pi_s(\lambda_v, \nu_v)$ and a non-tempered representation $\pi_n(\lambda_v, \nu_v)$ of $G'(F_v)$. These constituents of the packet can be described (see [R1, §12.2]) as the unique subrepresentation and the corresponding (Langlands) quotient representation of the induction of the character of $B(F_v)$ which is trivial on the unipotent subgroup and given on $T(F_v)$ by:

$$(\bar{\alpha}, \beta, \alpha^{-1}) \mapsto \lambda_v(\bar{\alpha})|\alpha|_{M_v}^{3/2}\nu_v(\beta), \text{ where } \alpha \in M_v^\times, \beta \in M_v^1. \tag{12}$$

If one considers unitary induction, then one has to divide the above character by the square root of the modular character of $B(F_v)$, that is to say by $(\bar{\alpha}, \beta, \alpha^{-1}) \mapsto |\alpha|_{M_v}$.

At any finite place $v$ of $F$ which splits in $M$, $G_v \simeq G'_v$ also splits and is isomorphic to $\mathrm{GL}(3, F_v)$. The local Arthur packet $\Pi'(\lambda_v, \nu_v)$ has a unique element $\pi_n(\lambda_v, \nu_v)$ which is induced from the character:

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$h_2$}} & * \\ & & * \\ \hline 0 & 0 & h_1 \end{array}\right) \mapsto \lambda_v(\det(h_2))|\det(h_2)|_v^{3/2}\nu_v(h_1)$$

of the maximal parabolic of type $(2,1)$ in $\mathrm{GL}(3, F_v)$ (see [R2, §1]).

For almost all $v$, $\pi_n(\lambda_v, \nu_v)$ is necessarily unramified. We set

$$\Pi'(\lambda, \nu) = \left\{\otimes_v \pi_v | \pi_v \in \Pi'(\lambda_v, \nu_v) \text{ for all } v \text{ , and } \pi_v \simeq \pi_n(\lambda_v, \nu_v) \text{ for almost all } v\right\}.$$

Recall that a CM type $\Phi$ on $M$ is the choice, for each Archimedean place $v$ of $F$, of an isomorphism $M \otimes_{F,v} \mathbb{R} \simeq \mathbb{C}$.

**Definition 3.1.** Let $\Xi$ denote the set of pairs $(\lambda, \nu)$ where $\lambda$ is a unitary Hecke character of $M$ whose restriction to $F$ is $\omega$, and $\nu$ is a unitary character of $\mathbb{A}_M^1/M^1$, such that

$$\lambda_\infty(z) = \prod_{v\in\Phi} \frac{\bar{z}_v}{|z_v|} \text{ , for all } z \in M_\infty \text{ and} \tag{13}$$

$$\nu_\infty(z) = \prod_{v\in\Phi} z_v \text{ , for all } z \in M_\infty^1,$$

for some CM type $\Phi$ on $M$.

**Theorem 3.2** (Rogawski [R1, R2])**.** (i) *For every $(\lambda, \nu) \in \Xi$, $\Pi'(\lambda, \nu)$ is a global Arthur packet for $G'$ such that for all infinite $v$, $\pi_n(\lambda_v, \nu_v) = \pi^+$ or $\pi^-$.*

(ii) *$\Pi'(\lambda, \nu)$ can be transferred to an Arthur packet $\Pi(\lambda, \nu)$ on $G$ such that $\Pi(\lambda_v, \nu_v) = \{\mathbb{1}\}$ at all the Archimedean places $v \neq \iota$, and $\Pi(\lambda_v, \nu_v) = \Pi'(\lambda_v, \nu_v)$ at the remaining places.*

(iii) *Denote by $W(\lambda\nu_M) \in \{\pm 1\}$ the root number of Hecke character $\lambda\nu_M$, where $\nu_M(z) = \nu(\bar{z}/z)$ for $z \in \mathbb{A}_M^\times$, and by $s(\pi)$ the number of finite places $v$ such that $\pi_v \simeq \pi_s(\lambda_v, \nu_v)$. Then*

$$\pi \in \Pi(\lambda, \nu) \textit{ is automorphic if and only if } W(\lambda\nu_M) = (-1)^{d-1+s(\pi)}.$$

*Moreover, in this case the global multiplicity $m(\pi)$ is 1.*

(iv) *Any automorphic representation $\pi$ of $G(\mathbb{A}_F)$ such that $\pi_\iota \simeq \pi^\pm$ and $\pi_v = \mathbb{1}$ at all the Archimedean places $v \neq \iota$, belongs to $\Pi(\lambda, \nu)$ for some $(\lambda, \nu) \in \Xi$.*

*Proof.* Let $H = \mathrm{U}(2) \times \mathrm{U}(1)$ be the unique elliptic endoscopic group, shared by $G'$ and all its inner forms over $F$. The embedding of $L$-groups ${}^LH \hookrightarrow {}^LG = {}^LG'$ depends on the choice of a Hecke character $\mu$ of $M$, whose restriction to $F$ is $\omega$, and allows one to transfer discrete $L$-packets on $H$ to automorphic $L$-packets on $G$ (see [R2, §13.3]). The character

$\mu$ being fixed, any pair of characters $(\lambda, \nu) \in \Xi$ uniquely determines a (one-dimensional) character of $H$, whose endoscopic transfer is $\Pi'(\lambda, \nu)$ (see [R2, §1]).

Denote by $W_F$ (resp. $W_M$) the global Weil group of $F$ (resp. $M$). By *loc.cit.*, the restriction to $W_M$ of the global Arthur parameter

$$W_F \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G = \mathrm{GL}(3, \mathbb{C}) \rtimes \mathrm{Gal}(M/F)$$

of $\Pi'(\lambda, \nu)$ is given by the 3-dimensional representation $(\lambda \otimes \mathrm{St}) \oplus (\nu_M \otimes \mathbb{1})$, where St (resp. $\mathbb{1}$) is the standard 2-dimensional (resp. trivial) representation of $\mathrm{SL}(2, \mathbb{C})$. By [La, p.62] the restrictions to $\mathbb{C}^\times$ of the Langlands parameters of $\pi^+$ and $\pi^-$ are given by $z \mapsto \begin{pmatrix} \bar{z} & 0 & 0 \\ 0 & z/\bar{z} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}$ and its complex conjugate, hence for every Archimedean place $v$ one has $\pi_n(\lambda_v, \nu_v) = \pi^+$ or $\pi^-$, depending on the choice of isomorphism $M \otimes_{F,v} \mathbb{R} \simeq \mathbb{C}$ in the CM type $\Phi$. It follows that for every Archimedean place $v$, $\Pi'(\lambda_v, \nu_v)$ is a packet containing a discrete series representation of $G'_v$, and thus by [R1, §14.4], there will be a corresponding Arthur packet $\Pi(\lambda, \nu)$ of representations of $G(\mathbb{A}_F)$ such that at any Archimedean place $v \neq \iota$, $\Pi(\lambda_v, \nu_v)$ is a singleton consisting of a finite-dimensional representation of the compact real group $G(F_v) = \mathrm{U}(3)$. In the notation of [R2, p.397] the representations $\pi^+$ and $\pi^-$ have parameters $(r, s) = (1, -1)$ and $(r, s) = (0, 1)$, respectively, and hence, by the recipe on the same page, the highest weight of the associated finite-dimensional representation equals $(1, 0, -1)$. Therefore at every Archimedean $v \neq \iota$ we have $\Pi(\lambda_v, \nu_v) = \{\mathbb{1}\}$.

So far we have established (i) and (ii), while (iii) is the content of [R2, Theorem 1.1].

Conversely, any $\pi$ as in (iv) is discrete, hence belongs to an Arthur packet $\Pi$ on $G$, which can be transferred to an Arthur packet $\Pi'$ on $G'$ (see [R1, §14.4 and Proposition 14.6.2]). By definition, $\Pi_v = \Pi'_v$ at $v = \iota$ and at all the finite places $v$ (where, as noted earlier, $G_v = G'_v$). In particular $\pi^+$ or $\pi^-$ belongs to $\Pi_\iota = \Pi'_\iota$, hence $\Pi'$ arises by endoscopy from $H$, that is to say equals $\Pi'(\lambda, \nu)$ for some unitary Hecke character $\lambda$ of $M$ whose restriction to $F$ is $\omega$, and some unitary character of $\mathbb{A}_M^1/M^1$ (see [R1, Theorem 13.3.6]). Since $\Pi_v = \{\mathbb{1}\}$ for all the Archimedean places $v \neq \iota$, by the above mentioned recipe $\Pi(\lambda_v, \nu_v)$ contains either $\pi^+$ or $\pi^-$, implying that $(\lambda, \nu) \in \Xi$ (see Definition 3.1). □

3.2. **Irregularity of the connected components.** Let $K$ be an open compact subgroup of $G(\mathbb{A}_{F,f})$. Using Theorem 3.2(iii) one can transform (10) into the formula:

$$4q(Y_K) = \sum_{(\lambda,\nu)\in\Xi} \sum_{\pi\in\Pi(\lambda,\nu)} \dim(\pi_f^K)(W(\lambda\nu_M) + (-1)^{d-1+s(\pi)}). \tag{14}$$

We will now deduce a similar formula for the irregularity of the connected component of identity $Y_\Gamma$ of $Y_K$, where $\Gamma = G(F) \cap KG_\infty$.

Recall (see (7)) that $\pi_0(Y_K) \simeq \mathbb{A}_M^1/M^1 \det(K) M_\infty^1$, and denote by $\widehat{\pi_0(Y_K)}$ its (finite, abelian) group of characters. Consider the free action of $\widehat{\pi_0(Y_K)}$ on the set $\Xi$ given by

$(\chi,(\lambda,\nu)) \mapsto (\lambda\chi_M^{-1},\nu\chi)$, and denote by $\Xi/\widehat{\pi_0(Y_K)}$ the quotient set. Since for any $\pi \in \Pi(\lambda,\nu)$ and any $\chi \in \widehat{\pi_0(Y_K)}$ one has $\pi\otimes\chi \in \Pi(\lambda\chi_M^{-1},\nu\chi)$, the group $\widehat{\pi_0(Y_K)}$ acts freely on the set of automorphic representations contributing to $q(Y_K)$. Moreover this action preserves $\lambda\nu_M$, $s(\pi)$ and the dimension of $\pi_f^K$. Hence, in the notations of (8), for any $1 \le i \le h$, the image of the composite map

$$\mathrm{H}^1(\mathrm{Lie}(G_\infty),K_\infty;\pi_\infty)\otimes\pi_f^K \to \mathrm{H}^1(Y_K,\mathbb{C}) \to \mathrm{H}^1(Y_{\Gamma_i},\mathbb{C}),$$

where the first map comes from (9) and the second from the inclusion $Y_{\Gamma_i} \subset Y_K$, does not change when replacing $\pi$ by $\pi\otimes\chi$ for any $\chi \in \widehat{\pi_0(Y_K)}$.

It follows that $q(Y_{\Gamma_i}) \le \frac{1}{h}q(Y_K)$ for all $1 \le i \le h$. Since $\sum_{i=1}^h q(Y_{\Gamma_i}) = q(Y_K)$, we deduce that $q(Y_{\Gamma_i}) = \frac{1}{h}q(Y_K)$ for all $1 \le i \le h$. This establishes the formula:

$$4q(Y_\Gamma) = \sum_{(\lambda,\nu)\in\Xi/\widehat{\pi_0(Y_K)}} \ \sum_{\pi\in\Pi(\lambda,\nu)} \dim(\pi_f^K)(W(\lambda\nu_M)+(-1)^{d-1+s(\pi)}). \tag{15}$$

This formula shows the importance of calculating $\dim(\pi_f^K)$ which, when $K$ is of the form $\prod_v K_v$ with $v$ running over all the finite places of $F$, can be reduced to a local computation of $\dim(\pi_v^{K_v})$. This will be taken up in the following section at places $v$ where $K_v$ is not the hyperspecial maximal compact subgroup $K_v^0$.

3.3. **Levels of induced representations.** Let $\mathfrak{p}$ be a prime of $F$ divisible by a unique prime $\mathfrak{P}$ of $M$ and let $\mathbb{F}_q$ be the residue field $\mathfrak{o}/\mathfrak{p}$. In this section we exhibit open compact subgroups $K_\mathfrak{p}$ of $G(F_\mathfrak{p})$ for which $\pi_n(\lambda_\mathfrak{p},\nu_\mathfrak{p})$ (resp. $\pi_s(\lambda_\mathfrak{p},\nu_\mathfrak{p})$) admit a non-zero $K_\mathfrak{p}$-invariant subspace, and compute in some cases the exact dimension of this space.

For every integer $m \ge 1$, we define the open compact subgroup $K(\mathfrak{P}^m)$ (resp. $K_0(\mathfrak{P}^m)$, resp. $K_1(\mathfrak{P}^m)$) of $G(F_\mathfrak{p})$ as the kernel (resp. the inverse image of upper triangular, resp. upper unipotent, matrices) of the composite homomorphism:

$$G(\mathfrak{o}_\mathfrak{p}) \hookrightarrow \mathrm{GL}(3,\mathfrak{O}_\mathfrak{P}) \to \mathrm{GL}(3,\mathfrak{O}/\mathfrak{P}^m). \tag{16}$$

**Lemma 3.3.** *Let $m \in \mathbb{Z}_{>0}$ be such that the character (12) is trivial on $K_1(\mathfrak{P}^m)\cap T(F_\mathfrak{p})$. Then both $\pi_n(\lambda_\mathfrak{p},\nu_\mathfrak{p})$ and $\pi_s(\lambda_\mathfrak{p},\nu_\mathfrak{p})$ have non-zero fixed vectors under $K_1(\mathfrak{P}^m)$.*

*Proof.* Let $J$ denote the Jacquet functor sending admissible $G(F_\mathfrak{p})$-representations to admissible $T(F_\mathfrak{p})$-representations. The Jacquet functor is exact and its basic properties imply:

$$\begin{aligned} J(\pi_s(\lambda_\mathfrak{p},\nu_\mathfrak{p})) &: (\bar\alpha,\beta,\alpha^{-1}) \mapsto \lambda_\mathfrak{p}(\bar\alpha)\nu_\mathfrak{p}(\beta)|\alpha|_{M_\mathfrak{p}}^{3/2} = \lambda_\mathfrak{p}(\bar\alpha)\nu_\mathfrak{p}(\beta)|\alpha|_{M_\mathfrak{p}}^{1/2}\cdot|\alpha|_{M_\mathfrak{p}},\\ J(\pi_n(\lambda_\mathfrak{p},\nu_\mathfrak{p})) &: (\bar\alpha,\beta,\alpha^{-1}) \mapsto \lambda_\mathfrak{p}(\bar\alpha)\nu_p(\beta)|\alpha|_{M_\mathfrak{p}}^{1/2} = \lambda_\mathfrak{p}(\alpha^{-1})\nu_\mathfrak{p}(\beta)|\alpha|_{M_\mathfrak{p}}^{-1/2}\cdot|\alpha|_{M_\mathfrak{p}}. \end{aligned} \tag{17}$$

One knows that $K_1(\mathfrak{P}^m)$ admits an Iwahori decomposition:

$$K_1(\mathfrak{P}^m) = (K_1(\mathfrak{P}^m)\cap N(F_\mathfrak{p}))\cdot(K_1(\mathfrak{P}^m)\cap T(F_\mathfrak{p}))\cdot(K_1(\mathfrak{P}^m)\cap \bar N(F_\mathfrak{p})),$$

where $N(F_{\mathfrak{p}})$ (resp. $\bar{N}(F_{\mathfrak{p}})$) denotes the unipotent of the standard (resp. opposite) Borel containing $T(F_{\mathfrak{p}})$. This is proved for the principal congruence subgroup $K(\mathfrak{P}^m)$ in [C, Proposition 1.4.4] and the extension to $K_1(\mathfrak{P}^m)$ is straightforward. Now by the proof of [C, Proposition 3.3.6], given any admissible $G(F_{\mathfrak{p}})$-representation $V$, one has a canonical surjection:

$$V^{K_1(\mathfrak{P}^m)} \twoheadrightarrow J(V)^{K_1(\mathfrak{P}^m)\cap T(F_{\mathfrak{p}})}.$$

Since both characters in (17) are trivial on $K_1(\mathfrak{P}^m) \cap T(F_{\mathfrak{p}})$, the claim follows. $\square$

**Lemma 3.4.** *Suppose that $\mathfrak{p}$ is inert in $M$ and that $(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})$ is unramified. Then the dimension of the $K_0(\mathfrak{p})$-fixed subspace of both $\pi_s(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})$ and $\pi_n(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})$ is 1. Moreover the dimension of the $K(\mathfrak{p})$-fixed subspace of $\pi_s(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})$ (resp. $\pi_n(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})$) is $q^3$ (resp. 1).*

*Proof.* Since $(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})$ is unramified, restriction to the standard hyperspecial maximal compact subgroup $K_{\mathfrak{p}}^0$ of $G(F_{\mathfrak{p}})$ yields, by Iwasawa decomposition $G(F_{\mathfrak{p}}) = B(F_{\mathfrak{p}}) \cdot K_{\mathfrak{p}}^0$, the following exact sequence:

$$0 \to \pi_s(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})_{|K_{\mathfrak{p}}^0} \to \mathrm{Ind}_{B(F_{\mathfrak{p}})\cap K_{\mathfrak{p}}^0}^{K_{\mathfrak{p}}^0}(\mathbb{1}) \to \pi_n(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})_{|K_{\mathfrak{p}}^0} \to 0.$$

The subspace of $K(\mathfrak{p})$-invariant vectors in $\mathrm{Ind}_{B(F_{\mathfrak{p}})\cap K_{\mathfrak{p}}^0}^{K_{\mathfrak{p}}^0}(\mathbb{1})$ identifies naturally with the space of $\mathbb{C}$-valued functions on the set:

$$\left(B(F_{\mathfrak{p}}) \cap K_{\mathfrak{p}}^0\right) \backslash K_{\mathfrak{p}}^0 / K(\mathfrak{p}) \simeq B(\mathbb{F}_q)\backslash G(\mathbb{F}_q),$$

on which $K_{\mathfrak{p}}^0/K(\mathfrak{p}) = G(\mathbb{F}_q)$ acts by right translation. By the Iwahori decomposition, since $G(\mathbb{F}_q)$ has rank 1, the representation $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\mathbb{1})$ has exactly two irreducible constituents which are the trivial representation and the Steinberg representation, implying that both $\pi_n(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})^{K_0(\mathfrak{p})}$ and $\pi_s(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})^{K_0(\mathfrak{p})}$ are one-dimensional. Since $\pi_s(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})^{K_{\mathfrak{p}}^0} = 0$, it follows that $\pi_n(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})^{K(\mathfrak{p})}$ (resp. $\pi_s(\lambda_{\mathfrak{p}}, \nu_{\mathfrak{p}})^{K(\mathfrak{p})}$ ) is isomorphic to the trivial (resp. Steinberg) representation of $G(\mathbb{F}_q)$, hence its dimension equals 1 (resp. $q^3$). $\square$

3.4. **Surfaces with positive irregularity.** The existence of Hecke characters $\lambda$ of $M$ satisfying (13) goes back to Chevalley and Weil. We will show that there are still such characters if one further imposes their restriction to $F$ to be $\omega$.

**Lemma 3.5.** *For any CM extension $M/F$ and any CM type $\Phi$ on $M$, there exist Hecke characters $\lambda$ of $M$ whose restriction to $F$ equals $\omega$, such that $\lambda_\infty(z) = \prod_{v\in\Phi} \frac{\bar{z}_v}{|z_v|}$ for all $z \in M_\infty^\times$.*

*Proof.* Since $M$ is totally imaginary, $\lambda_\infty$ and $\omega$ agree on $F_\infty^\times$, hence there is a character $\lambda_0$ of $\mathbb{A}_F^\times M_\infty^\times$ extending both.

We show now that $\lambda_0$ can be extended to a Hecke character $\lambda$ of $M$, which will obviously satisfy the assumptions of the lemma. Since $M/F$ is totally imaginary, $\mathfrak{o}^{\times 2}$ has finite index in $\mathfrak{O}^\times$. By [Ch, Théorème 1] there exists an open compact subgroup $U$ of $\mathbb{A}_{M,f}^\times$ such that $U \cap \mathfrak{O}^\times \subset \mathfrak{o}^{\times 2}$. We may, and we will, assume that $U$ is contained in the congruence subgroup whose level is the relative different of $M/F$ and by replacing $U$ by $U \cap \bar{U}$ we can further assume that $U = \bar{U}$. Since the Artin conductor of $\omega$ is the relative discriminant of $M/F$, it follows that $\omega$ is trivial on $U \cap \mathbb{A}_{F,f}^\times$. Hence one can extend $\lambda_0$ to a character of $\mathbb{A}_F^\times U M_\infty^\times$ by letting it be trivial on $U$.

Suppose we knew that

$$M^\times \cap \mathbb{A}_F^\times U M_\infty^\times = F^\times. \tag{18}$$

Then there is a unique character of $M^\times \mathbb{A}_F^\times U M_\infty^\times$ extending both $\lambda_0$ and the trivial character of $M^\times U$. Since $\mathbb{A}_M^\times / M^\times \mathbb{A}_F^\times U M_\infty^\times$ is a finite abelian (idele class) group, the character above can be further extended to a character $\lambda$ of $\mathbb{A}_M^\times / M^\times$, and any such extension has the desired properties.

It remains to prove (18). Let $x \in M^\times \cap \mathbb{A}_F^\times U M_\infty^\times$. Then

$$\bar{x}/x \in M^1 \cap U M_\infty^1 = \mathfrak{O}^\times \cap U M_\infty^1 \subset \mathfrak{o}^{\times 2}.$$

Since $F^{\times 2} \cap M^1 = \{1\}$, we have $x = \bar{x} \in F^\times$. □

**Proposition 3.6.** *Fix any Hecke character $\lambda$ of $M$ satisfying* (13) *whose restriction to $F$ is $\omega$. Let $\mathfrak{p}$ be a prime of $F$ which splits in $M$ and is relatively prime to the conductor $\mathfrak{C}$ of $\lambda$. If $W(\lambda^3) = (-1)^d$ we choose a prime $\mathfrak{q}$ of $F$ which does not split in $M$; if not, we take $\mathfrak{q} = \mathfrak{o}$. Then $q(Y_{\Gamma_1(\mathfrak{C}) \cap \Gamma_0(\mathfrak{pq})}) > 2$.*

*Proof.* Let $K = K_1(\mathfrak{C}) \cap K_0(\mathfrak{pq})$, so that $\Gamma = G(F) \cap K K_\infty$. Let $\Pi(\lambda, \lambda^{-1}_{|M^1})$ be the global Arthur packet on $G$ associated to $\lambda$. Let $\pi = \otimes_v \pi_v \in \Pi(\lambda, \lambda^{-1}_{|M^1})$ be such that $\pi_\iota = \pi^+$, $\pi_v = \mathbb{1}$ for every infinite place $v \neq \iota$, $\pi_v = \pi_{n,v}$ for every finite $v \neq \mathfrak{q}$, and finally if $W(\lambda^3) = (-1)^d$ then $\pi_\mathfrak{q} = \pi_{s,\mathfrak{q}}$. By Theorem 3.2(iii), $\pi$ is automorphic, and by Lemma 3.3, we have $\pi_{n,v}^{K_v} \neq 0$ for all finite places $v \neq \mathfrak{p}, \mathfrak{q}$. Moreover, if $\mathfrak{q} \neq \mathfrak{o}$, then $\pi_{s,\mathfrak{q}}^{K_\mathfrak{q}} \neq 0$ by Lemma 3.3 (resp. Lemma 3.4 ) if $\mathfrak{q}$ divides (resp. does not divide) $\mathfrak{C}$. Finally (11) implies that $\dim(\pi_\mathfrak{p}^{K_0(\mathfrak{p})}) \geq 3$, hence $q(Y_\Gamma) \geq 3$ by (15) as claimed. □

**Remark 3.7.** Since restriction of $\lambda$ to $\mathbb{A}_F^\times$ equals $\omega$, its conductor $\mathfrak{C}$ is divisible by the different of $M/F$. Hence, unless $M/F$ is unramified everywhere, one might take as $\mathfrak{q}$ a place where $M/F$ is ramified and Proposition 3.6 applies then to $K = K_1(\mathfrak{C}) \cap K_0(\mathfrak{p})$. Given a totally real number field $F$, there exists a totally imaginary quadratic extension $M/F$ unramified everywhere if and only if all the units in $F$ have norm 1.

For the rest of this section we assume that $F = \mathbb{Q}$, so that $G$ is quasi-split.

**Proposition 3.8.** *Any $\Gamma$ as in Theorem 0.3 is neat and $q(Y_\Gamma) > 2$.*

*Proof.* There exists an open compact subgroup $K$ of $G(\mathbb{A}_{\mathbb{Q},f})$ such that $\Gamma = G(\mathbb{Q}) \cap KG(\mathbb{R})$.

We claim that there exists a Hecke character $\lambda$ of $M$ of conductor $\mathfrak{D}$ satisfying (13), whose restriction to $F$ equals $\omega$. If $D \neq 3$ is odd or if 8 divides $D$ such characters, called canonical, are proved to exists by Rohrlich [Ro]. If $D > 4$ is even but not divisible by 8, then by Yang (see [Ya, p.88]) there are such characters, called the *simplest*. Finally for $D = 3$ (resp. D=4) the claim follows from the existence of a CM elliptic curve over $\mathbb{Q}$ of conductor 27 (resp. 32).

By definition, $(\lambda, \lambda^{-1}_{|M^1}) \in \Xi$ and is trivial on $K_1(\mathfrak{D}) \cap T(\mathbb{A}_{\mathbb{Q},f})$. Then Lemma 3.3 implies that:

$$\pi_f^{K_1(\mathfrak{D})} \neq 0, \text{ for all } \pi \in \Pi(\lambda, \lambda^{-1}_{|M^1}). \tag{19}$$

In case (ii) of Theorem 0.3 where $\Gamma = \Gamma(N) \cap \Gamma_1(\mathfrak{D})$ we fix a prime $p$ dividing $D$ and $\pi = \otimes_v \pi_v \in \Pi(\lambda, \lambda^{-1}_{|M^1})$ such that $\pi_v = \pi_n(\lambda_v, \lambda^{-1}_{|M^1_v})$ for all $v \neq p, N$, $\pi_N = \pi_s(\lambda_N, \lambda^{-1}_{|M^1_N})$ and

$$\pi_p = \begin{cases} \pi_n(\lambda_p, \lambda^{-1}_{|M^1_p}) \text{ , if } W(\lambda^3) = -1, \\ \pi_s(\lambda_p, \lambda^{-1}_{|M^1_p}) \text{ , if } W(\lambda^3) = 1. \end{cases}$$

Since $\Gamma(N)$ is neat by Lemma 1.4, we can apply (15) which, when combined with Lemma 3.4, yields

$$q(Y_{\Gamma(N)\cap\Gamma_1(\mathfrak{D})}) \geq \dim(\pi_N^{K(N)}) \geq N^3 \geq 3.$$

We now turn to case (i) and suppose first that $M$ has class number $h \geq 3$. For any class character $\xi$ one has $(\lambda\xi, \lambda^{-1}_{|M^1}) \in \Xi$ giving $h$ pairwise distinct elements in $\Xi/\pi_0(\widehat{Y_{K_1(\mathfrak{D})}})$. Fix a prime $p$ dividing $D$ and consider $\pi = \otimes_v \pi_v \in \Pi(\lambda\xi, \lambda^{-1}_{|M^1})$ such that $\pi_v = \pi_n(\lambda_v\xi_v, \lambda^{-1}_{|M^1_v})$ for all $v \neq p$ and

$$\pi_p = \begin{cases} \pi_n(\lambda_p\xi_p, \lambda^{-1}_{|M^1_p}) \text{ , if } W(\lambda^3) = 1, \\ \pi_s(\lambda_p\xi_p, \lambda^{-1}_{|M^1_p}) \text{ , if } W(\lambda^3) = -1. \end{cases}$$

Since $\Gamma_1(\mathfrak{D})$ is neat by Lemma 1.5, one can apply (15) which combined with (19) yields $q(Y_{\Gamma_1(\mathfrak{D})}) \geq h \geq 3$.

If $M$ is one of the 18 imaginary quadratic fields of class number 2, then its fundamental discriminant $D$ has (exactly) two distinct prime divisors $p < q$. For each character $\lambda$ on $M$ as above, consider $\pi \in \Pi(\lambda, \lambda^{-1}_{|M^1})$ such that $\pi_v = \pi_n(\lambda_v, \lambda^{-1}_{|M^1_v})$ for all $v \neq p, q$ and

$$(\pi_p, \pi_q) = \begin{cases} (\pi_n(\lambda_p, \lambda^{-1}_{|M^1_p}), \pi_n(\lambda_q, \lambda^{-1}_{|M^1_q})) \text{ or } (\pi_s(\lambda_p, \lambda^{-1}_{|M^1_p}), \pi_s(\lambda_q, \lambda^{-1}_{|M^1_q})) \text{ , if } W(\lambda^3) = 1, \\ (\pi_n(\lambda_p, \lambda^{-1}_{|M^1_p}), \pi_s(\lambda_q, \lambda^{-1}_{|M^1_q})) \text{ or } (\pi_s(\lambda_p, \lambda^{-1}_{|M^1_p}), \pi_n(\lambda_q, \lambda^{-1}_{|M^1_q})) \text{ , if } W(\lambda^3) = -1. \end{cases}$$

If $D \neq 24$ then $\Gamma_1(\mathfrak{D})$ is neat by Lemma 1.5 and (15) implies that $q(Y_{\Gamma_1(\mathfrak{D})}) \geq 2 \cdot 2 = 4$. If $D = 24$ then $\Gamma(\mathfrak{D})$ is neat by Lemma 1.4, since 4 divides $\mathfrak{D}$, and again $q(Y_{\Gamma(\mathfrak{D})}) \geq 4$.

Finally, we consider the nine imaginary quadratic fields of class number 1.

For $D \in \{7, 11, 19, 43, 67, 163\}$ there is a unique character $\lambda$ as in the beginning of the proof. Any character of $(1+\sqrt{-D}\mathfrak{O}/1+D\mathfrak{O}) \simeq \mathbb{Z}/D\mathbb{Z}$ lifts to a finite order Hecke character $\xi$ of $M$ with trivial restriction to $\mathbb{Q}$, hence $(\lambda\xi, \lambda^{-1}_{|M^1}) \in \Xi$. Let $\pi = \otimes_v \pi_v \in \Pi(\lambda\xi, \lambda^{-1}_{|M^1})$ be such that $\pi_v = \pi_n(\lambda_v\xi_v, \lambda^{-1}_{|M^1_v})$ for all $v \neq D$ and

$$\pi_D = \begin{cases} \pi_n(\lambda_D\xi_D, \lambda^{-1}_{|M^1_D}) \text{ , if } W(\lambda^3) = 1, \\ \pi_s(\lambda_D\xi_D, \lambda^{-1}_{|M^1_D}) \text{ , if } W(\lambda^3) = -1. \end{cases}$$

Since $\Gamma(D)$ is neat by Lemma 1.4, by (15) we get $q(Y_{\Gamma(D)}) \geq D \cdot \dim(\pi_D^{K(D)}) \geq D$.

For $D = 3$ the same argument with $D^2$ instead of $D$, implies that $q(Y_{\Gamma(9\mathfrak{O})}) \geq 3$.

For $D = 4$ (resp. $D = 8$) the group $\Gamma(8\mathfrak{O})$ (resp. $\Gamma(2\sqrt{-8}\mathfrak{O})$) is neat by Lemma 1.4 and it is an exercise on idele class groups to show that there are at least three Hecke characters of $M$ satisfying (13) whose restriction to $\mathbb{Q}$ is $\omega$, and whose conductor divides 8 (resp. $2\sqrt{-8}$). It follows then from (15) and (19) that for $D = 4$ (resp. $D = 8$) one has $q(Y_{\Gamma(8\mathfrak{O})}) \geq 3$ (resp. $q(Y_{\Gamma(2\sqrt{-8}\mathfrak{O})}) \geq 3$). □

**Remark 3.9.** The computation of the smallest level $K$ for which there exists an automorphic representation $\pi \in \Pi(\lambda, \nu)$ such that $\pi_f^K \neq 0$ is analyzed in detail in [DR]. In particular, if $\lambda$ is a canonical character, we check that the level subgroup at any $p$ dividing $D$ is precisely the one conjectured by B. Gross, namely the index 2 subgroup of the maximal parahoric subgroup with reductive quotient PGL(2).

## 4. The Albanese map and Mordellicity

A major ingredient in the proof of our theorems is the Mordell-Lang conjecture for abelian varieties in characteristic zero, established by Faltings [F2] using some earlier work of himself [F1] and Vojta [V] (see Mazur's detailed account [M]).

**Theorem 4.1** (Mordell-Lang conjecture : theorem of Faltings)**.** *Suppose $A$ is an abelian variety over $\mathbb{C}$ and $Z \subset A$ a closed subvariety. Then for any finitely generated field extension $k$ of $\mathbb{Q}$ over which $Z \subset A$ is defined, the set $Z(k)$ is contained in a union of finitely many translates of abelian subvarieties of $A$, each of which is defined over $k$ and contained in $Z$.*

The following corollary was proved in Moriwaki [Mo, Theorem 1.1]. He stated it for number fields, but the proof is the same for finitely generated fields over $\mathbb{Q}$.

**Corollary 4.2.** *Let $X$ be a connected smooth projective variety over $\mathbb{C}$ which does not admit a dominant map to its Albanese variety. Then for any finitely generated field extension $k$ of $\mathbb{Q}$ over which $X$ is defined, the set $X(k)$ is not Zariski dense in $X$.*

*Proof.* The conclusion is obvious if $X(k)$ is empty so we may choose a point of $X(k)$ to define the Albanese map over $k$:

$$j : X \to \mathrm{Alb}(X).$$

Applying Theorem 4.1 to the closed subvariety $Z = j(X)$ of $\mathrm{Alb}(X)$ we get a finite number, say $m \geq 1$, of translates $Z_i$ of abelian subvarieties of $\mathrm{Alb}(X)$ defined over $k$ and such that

$$Z(k) \subset \bigcup_{i=1}^{m} Z_i(k) \quad \text{and} \quad Z_i \subset Z.$$

Since $j$ is defined over $k$, each $k$-rational point of $X$ is contained in $j^{-1}(Z_i)$ for some $i$. If $j^{-1}(Z_i)$ were not a proper closed subvariety of $X$, the universal property of the Albanese map would imply that $Z = Z_i = \mathrm{Alb}(X)$, contradicting the assumption that $j$ is not dominant. □

**Proposition 4.3.** *For every arithmetic subgroup $\Gamma \subset G(F)$ there exists a finite cover of $Y_\Gamma$ whose points over any finitely generated field extension of $\mathbb{Q}$ are not Zariski dense, i.e., the Bombieri-Lang conjecture holds for that cover.*

*Proof.* Applying Corollary 2.2 recursively yields a finite index subgroup $\Gamma' \subset \Gamma$, which one can assume to be torsion free, such that $q(Y_{\Gamma'}) > n = \dim(Y_{\Gamma'})$. It suffices then to apply Corollary 4.2 to $Y_{\Gamma'}$ in the compact case and, in view of (4), to $X_{\Gamma'}$ in the non-compact case. □

*Proof of Theorem 0.1.* By Proposition 3.6, we have $q(Y_\Gamma) > 2$, hence $Y_\Gamma$ does not admit a dominant map to its Albanese variety. Moreover $Y_\Gamma$ is a geometrically irreducible smooth projective surface, hence by Corollary 4.2 $Y_\Gamma(k)$ is not Zariski dense in $Y_\Gamma$ for any finitely generated field extension $k$ of $\mathbb{Q}$ over which $Y_\Gamma$ is defined. If $Y_\Gamma(k)$ were infinite, then $Y_\Gamma$ would contain an irreducible curve $C$ defined over $k$ and such that $C(k)$ infinite. Since $C(k)$ is Zariski dense in $C$, the curve $C$ is geometrically irreducible and its geometric genus is at most 1 by Theorem 4.1 applied to the Albanese map of $C$. Taking a complex uniformization of $C$ would provide a non-constant holomorphic map from $\mathbb{C}$ to $Y_\Gamma$, which is impossible by Lemma 1.1(i) which we can apply as $\Gamma$ is torsion free. Therefore $Y_\Gamma$ is Mordellic. □

*Proof of Theorem 0.2.* The Lang locus of a quasi-projective irreducible variety $Z$ over a number field is defined as the Zariski closure of the union, over all number fields $k$, of irreducible components of positive dimension of the Zariski closure of $Z(k)$. It is clear that $Z$ is arithmetically Mordellic if and only if its Lang locus is empty. The main theorem in

[UY] asserts that, for $\Gamma$ neat and sufficiently small, the Lang locus of $Y_\Gamma^*$ is either empty or everything.

By Corollary 2.2 one can assume by further shrinking $\Gamma$ that $q(Y_\Gamma) > n$, and by (4) we also have $q(X_\Gamma) > n$ for $X_\Gamma$ a smooth toroidal compactification of $Y_\Gamma$. By Corollary 4.2 the Lang locus of $X_\Gamma$ is not everything, which forces the Lang locus of $Y_\Gamma^*$ to be empty. □

*Proof of Theorem 0.3.* Let us first show that $X_\Gamma$ is of general type, hence its canonical divisor $\mathcal{K}_X$ is big in the sense of [N, Definition 1.1]. Note that just like irregularity, the Kodaira dimension cannot decrease when going to a finite covering. By Holzapfel [H2, Theorem 5.4.15] and Feustel [Fe] the surface $X_{\Gamma_1(\mathfrak{D})}$ is of general type for all

$$D \notin \{3, 4, 7, 8, 11, 15, 19, 20, 23, 24, 31, 39, 47, 71\}.$$

Also by [H1, Proposition 4.13], $X_{\Gamma(N)}$ is of general type for all integers $N > 2$, with the possible exceptions of $N = 3$ and $N = 4$ when $D = 4$, implying in particular that $X_{\Gamma(\mathfrak{D}^2)}$ is of general type for $D \in \{3, 4, 7, 8, 11, 19\}$. Finally, the argument from *loc. cit.* transports in a straight forward way to the case when the level is an ideal of $\mathfrak{O}$, yielding that the remaining varieties $X_{\Gamma(\mathfrak{D})}$, $D \in \{15, 20, 23, 24, 31, 39, 47, 71\}$, are of general type as well. See [D] where this has been carried out (it should be noted that his argument works also when $D = 24$).

If $g = \sum_{i,j=1}^{2} g_{i\bar{j}} dz_i d\bar{z}_j$ denotes the Bergman metric of $\mathcal{H}^2_{\mathbb{C}}$ viewed as the unit ball $\{z = (z_1, z_2) \in \mathbb{C}^2\, ,\, |z| < 1\}$, normalized by requiring that

$$\mathrm{Ric}(g) = \sum_{i,j=1}^{2} -\frac{\partial^2 \log(g_{1\bar{1}} g_{2\bar{2}} - g_{2\bar{1}} g_{1\bar{2}})}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j = -g,$$

then the holomorphic sectional curvature is constant and equals $-4/3$ (see [GKK, §3.3]), where $g_{i\bar{j}} = \frac{3\left((1-|z|^2)\delta_{ij} + \bar{z}_i z_j\right)}{(1-|z|^2)^2}$.

By (4) and Proposition 3.8 we have that $\Gamma$ is neat and $q(X_\Gamma) = q(Y_\Gamma) > 2$. Corollary 4.2 then implies that $X_\Gamma(k)$ is not Zariski dense in $X_\Gamma$ for any finitely generated field extension $k$ of $\mathbb{Q}$ over which $X_\Gamma$ is defined. If $X_\Gamma(k)$ is infinite, arguing as in the proof of Theorem 0.1 shows that $X_\Gamma$ contains a geometrically irreducible curve $C$ whose geometric genus is at most one. Now applying a result of Nadel [N, Theorem 2.1] with $\gamma = 1$ (so that $-\gamma \geq -4/3$), we see that the bigness of $\mathcal{K}_X$ implies that $C$ is contained in the compactifying divisor, which is a finite union of elliptic curves indexed by the cusps. It follows that $Y_\Gamma^*(k)$ is finite and that $X_\Gamma$ does not contain any rational curves at all, let alone just those of self intersection $-1$, hence it is a minimal surface of general type. □

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